



TITLE:

テータ級数のトレースの計算へのWeil表現の応用(代数群上の保型形式)

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CITATION:

Kume, Takahiro. テータ級数のトレースの計算へのWeil表現の応用(代数群上の保型形式). 数理解析研究所講究録 1996, 965: 71-86

ISSUE DATE:

1996-08

URL:

<http://hdl.handle.net/2433/60592>

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テータ級数のトレースの計算への Weil 表現の応用

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1. はじめに

標題の講演は、講演者が作成中のプレプリント (英文) にもとづいている。そこで、このプレプリントの原稿に手を加えたものを講演のレジюмеとして以下に掲げる。その主な内容は

- 問題の説明 [§2]
- 定理 V と定理 L (このふたつが主結果) の記述 [§4.2]
- 定理 L_p (定理 L の局所化) の記述 [§5.2]
- 定理 L_p から主結果が従うことの証明 [§5.4]

である (定理 L_p 自体の証明は紙数の都合で割愛した)。

この講演ならびにプレプリントの内容について、京都大学の吉田敬之教授から多くの有益な助言を頂いた。講演者は吉田教授に対して心からの感謝の意をあらわしたい。

2. MOTIVATION

Using the Weil representation, we calculate a trace of a theta series associated with a lattice of a certain quadratic space over \mathbb{Q} in order to examine the relation of the space of Siegel modular forms and the space of such theta series.

Let us explain our problem in more detail. For simplicity, suppose n is an even positive integer. Let S be a rational symmetric matrices of size $2n$ such that the determinant of S is a square of nonzero rational numbers. Let $V = (\mathbb{Q}^{2n}, Q)$ be a positive definite regular quadratic space of rank $2n$ over \mathbb{Q} represented by S .

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For every integral lattice L of V , we define the theta series ϑ_L by the following formula:

$$\vartheta_L(\mathfrak{z}) = \sum_{\mathfrak{z} \in L^m} \exp\left(2\pi\sqrt{-1}\mathrm{tr}\left({}^t x S x\right)\right)$$

where \mathfrak{z} is an element of the Siegel upper half space of degree m . If the level of L divides a positive integer N , the theta series ϑ_L belongs to the space $\mathcal{M}_m(n, N)$ of Siegel modular forms of weight n , degree m and level N . Let $\theta_m(V, N)$ be the subspace of $\mathcal{M}_m(n, N)$ spanned by such theta series.

For any positive divisor N' of N we obtain the following inclusion:

$$\theta_m(V, N') \subset \theta_m(V, N) \cap \mathcal{M}_m(n, N').$$

But it is not obvious whether or not the equality holds in the above formula.

Problem V. *Is the following equality*

$$\theta_m(V, N') = \theta_m(V, N) \cap \mathcal{M}_m(n, N')$$

true?

To attack Problem V, we can use the global trace operator $T_{N, N'}^{(n)}$ which is defined as follows. For every $\varphi \in \mathcal{M}_m(n, N)$ put

$$T_{N, N'}^{(n)}(\varphi)(\mathfrak{z}) = \sum_{\gamma} \det(c\mathfrak{z} + d)^{-1} \varphi\left((a\mathfrak{z} + b)(c\mathfrak{z} + d)^{-1}\right)$$

($\mathfrak{z} \in H_m$) where $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ runs over a complete set of representative of $\Gamma_0^{(m)}(N) \setminus \Gamma_0^{(m)}(N')$.

Then we have $T_{N, N'}^{(n)}(\varphi) \in \mathcal{M}_m(n, N')$.

If we get

$$T_{N, N'}^{(n)}(\theta_m(V, N')) \subset \theta_m(V, N'),$$

we solve Problem V. Thus we reduce Problem V to Problem L stated below:

Problem L. *Does the global trace $T_{N, N'}^{(n)}(\vartheta_L)$ belongs to $\mathcal{M}_m(n, N')$ for every integral lattice L of V of level N ?*

These two problems are discussed in several papers [3, 4, 1]. The authors of these papers reduce Problem L to the case when degree m is greater than $2n$ by establishing relative commutation relations of the global trace operators and the Siegel's ϕ -operators.

In contrast to their global method, we try to transform Problem L to a p -adic analogue by means of the global and local Weil representation.

3. NOTATION

The following notation will be used throuout of this paper: Let m, n be positive integers. For an associative ring A with identity element we denote by A^\times the group of all invertible elements and by $\text{Mat}_{m,n}(A)$ the module of all $m \times n$ -matrices with all entries in A ; we put $A^m = \text{Mat}_{m,1}(A)$, $\text{Mat}_m(A) = \text{Mat}_{m,m}(A)$ for simplicity. The identity and zero elements of the ring $\text{Mat}_m(A)$ are denoted by 1_m and 0_m (when m needs to be stressed). The transpose of a matrix g is denoted by ${}^t g$. We denote by $\text{tr}(x)$ the trace of a square matrix x . Let J be an A -submodule of A . We denote by $\text{Sym}_m(J)$ the module of all $m \times m$ -symmetric matrix with all entries in J . If all entries of a matrix $g \in \text{Mat}_{m,n}(A)$ belongs to J , we write $g \equiv 0 \pmod{J}$. We put

$$\mathbb{T} = \{\zeta \in \mathbb{C} \mid |\zeta| = 1\},$$

$$e(c) = \exp(2\pi\sqrt{-1}c) \quad (c \in \mathbb{C}).$$

For any set E , $|E|$ means the cardinality of E . The characteristic function of a subset E' of E is denoted by $I_{E'}$. For every locally compact Hausdorff group X , we denote by $\mathcal{S}(X)$ the space of Schwartz-Bruhat functions on X .

Let ∞ and \mathfrak{h} be the infinite place and the set of all finite places of \mathbb{Q} , respectively. We identify the latter set \mathfrak{h} with the set of all rational primes. We denote by \mathbb{Q}_v the completion of \mathbb{Q} at v for any place v of \mathbb{Q} . Let \mathfrak{G} be an algebraic group defined over \mathbb{Q} . For any field k containing \mathbb{Q} , we denote by \mathfrak{G}_k the group of k -rational points of \mathfrak{G} and abbreviate $\mathfrak{G}_{\mathbb{Q}_v}$ to \mathfrak{G}_v for each place v of \mathbb{Q} . We define the adelization $\mathfrak{G}_{\mathbb{A}}$ of \mathfrak{G} and view $\mathfrak{G}_{\mathbb{Q}}$ and \mathfrak{G}_v as subgroups of $\mathfrak{G}_{\mathbb{A}}$ as usual. We then denote by \mathfrak{G}_{∞} and $\mathfrak{G}_{\mathfrak{h}}$ the infinite and

the finite part of \mathfrak{G}_A , respectively. For $g \in \mathfrak{G}_A$, we denote by g_v , g_∞ , and g_h its projections to \mathfrak{G}_v , \mathfrak{G}_∞ , \mathfrak{G}_h .

We denote by $G^{(m)}$ the symplectic group of genus m . For a commutative ring R with identity element, we assume that the group of all R -rational points of $G_R^{(m)}$ of $G^{(m)}$ is given explicitly by

$$G_R^{(m)} = \left\{ g \in \mathrm{GL}_{2m}(R) \mid {}^t g \begin{bmatrix} 0_m & 1_m \\ -1_m & 0_m \end{bmatrix} g = \begin{bmatrix} 0_m & 1_m \\ -1_m & 0_m \end{bmatrix} \right\}.$$

We usually denote every element g of $G_R^{(m)}$ as $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $m \times m$ -matrices a, b, c, d . Let \mathcal{H}_m be the Siegel upper half space of genus m . We define an action of $G_\infty^{(m)}$ on \mathcal{H}_m and the factors of automorphy $j(\cdot, \cdot)$ as follows:

$$g\mathfrak{z} = (a\mathfrak{z} + b)(c\mathfrak{z} + d)^{-1},$$

$$j(g, \mathfrak{z}) = (c\mathfrak{z} + d),$$

where $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_\infty^{(m)}$ and $\mathfrak{z} \in \mathcal{H}_m$. For any positive integer N , we define a congruence subgroup $\Gamma_0^{(m)}(N)$ by

$$\Gamma_0^{(m)}(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_{2m}(\mathbb{Z}) \cap G_\infty^{(m)} \mid c \equiv 0 \pmod{N} \right\}.$$

Let (n, N) be a pair of positive integers such that N is arbitrary if n is even or N divides 4 if n is odd. For such a pair (n, N) we define the action of $\Gamma_0^{(m)}(N)$ on the space of all holomorphic functions on \mathcal{H}_m as follows :

$$(f|_n \gamma)(\mathfrak{z}) = \chi_n(\gamma) j(\gamma, \mathfrak{z})^{-n} f(\gamma \mathfrak{z})$$

where f is a holomorphic function on \mathcal{H}_m , $\gamma \in \Gamma_0^{(m)}(N)$, $\mathfrak{z} \in \mathcal{H}_m$, and χ_n is the character of $\Gamma_0^{(m)}(N)$ given by

$$\chi_n(\gamma) = (\det a, (-1)^n)_{\mathbb{Q}_2}$$

$\left(\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0^{(m)}(N) \right)$. We denote by $\mathcal{M}_m(n, N)$ the space of all holomorphic functions on \mathcal{H}_m satisfying the condition

$$f|_n \gamma = f$$

for any element γ of $\Gamma_0^{(m)}(N)$.

Let F be a local field with the characteristic of $F \neq 2$. We denote by $(\cdot, \cdot)_F$ the Hilbert symbol of index 2 over F . Let $V = (V, Q)$ be a regular quadratic space of rank n over F . We denote by B the nondegenerated bilinear form associated with the quadratic form Q given by $B(x, y) = Q(x + y) - Q(x) - Q(y)$ ($x, y \in V$). For some basis $\{e_i\}$ of V and for some $n \times n$ -regular symmetric matrix S , we have $Q(\sum_i x_i e_i) = {}^t x S x$ for any $x = [x_i] \in F^n$. We put

$$\det V = \det S \pmod{(F^\times)^2} \in F^\times / (F^\times)^2$$

It is independent of the choice of $\{e_i\}$ and S . The Hasse symbol of V is denoted by $\varepsilon_F(V)$.

For some $a_i \in F^\times$ ($i = 1, \dots, n$) and some basis $\{f_i\}$ of V , we get

$$Q\left(\sum_i x_i f_i\right) = \sum_i a_i x_i^2$$

for any $x = [x_i] \in F^n$; in this case we obtain

$$\varepsilon_F(V) = \prod_{1 \leq i < j \leq n} (a_i, a_j)_F.$$

Furthermore assume F is non-archimedean. Let R , ϖ and q be the maximal compact subring, a prime and the module of F . For every lattice L of V , put

$$L^\vee = \left\{ x \in V \mid B(x, y) \in R \quad (\forall y \in L) \right\}.$$

Then, L^\vee is also a lattice of V . If L is integral ($Q(L) \subset R$) the R -module generated by $Q(L^\vee)$ in F can be written as $\varpi^{-l}R$ for some non negative integer l . This number is denoted by $\text{lev}_V(L)$. Represent the quadratic form Q as a symmetric matrix S' by taking

some R -basis of L . We denote by $\det L$ the element $\det S' \bmod (R^\times)^2$ of $F^\times/(R^\times)^2$. It is independent of the choice of R -basis and S' .

Let $V = (V, Q)$ be a regular quadratic space of rank n over \mathbb{Q} . For each place v of \mathbb{Q} , we denote by $V_v = (V_v, Q)$ the scalar extension of V over \mathbb{Q}_v as quadratic space. We put $\varepsilon_v(V) = \varepsilon_{\mathbb{Q}_v}(V)$. For every $p \in \mathfrak{h}$ and every lattice M of V_p , we put $\text{lev}_p(M) = \text{lev}_{V_p}(M)$. For every $p \in \mathfrak{h}$ and every lattice L of V , we put $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. This module L_p becomes a lattice of V_p . If L is an integral lattice of V ($Q(L) \subset \mathbb{Z}$), L_p is also an integral lattice of V_p for every $p \in \mathfrak{h}$ and $\text{lev}_p(L_p) = 0$ for almost all $p \in \mathfrak{h}$. We denote by $\text{level}(L)$ the integer $\prod_{p \in \mathfrak{h}} p^{\text{lev}_p}$. We define $\det V$ and $\det L$ as similarly as in the local field case.

4. MAIN RESULTS

4.1. Preliminaries. Let N, m, n be positive integers. Fix a nonsingular symmetric matrix $S \in \text{Sym}_{2n}(\mathbb{Q}) \cap \text{GL}_{2n}(\mathbb{Q})$. We obtain a regular quadratic space $V = (\mathbb{Q}^{2n}, Q)$ of rank $2n$ over \mathbb{Q} by $Q(x) = {}^t x S x$ ($x \in \mathbb{Q}^{2n}$). Let B be the nondegenerated bilinear form associated with the quadratic form Q determined by $B(x, y) = Q(x + y) - Q(x) - Q(y)$ ($x, y \in V$). Let X be the vector-space direct sum of m -copies of V . We identify this vector space X with $\text{Mat}_{2n, m}(F)$. For any $x \in X$, we write $x = (x_i)$ by column vectors x_i ($1 \leq i \leq m$).

For any integral lattice L of V , define the theta series associated with L by the following formula:

$$\vartheta_L(\mathfrak{z}) = \sum_{x \in L^m} e\left(\text{tr}({}^t x S x \mathfrak{z})\right)$$

($\mathfrak{z} \in \mathcal{H}_m$). It is well-known that if $\text{level}(L)$ divides N then

$$\vartheta_L \in \mathcal{M}_m(n, N).$$

Let $\Theta_m(V, N)$ be the subspace of $\mathcal{M}_m(n, N)$ generated by

$$\{\vartheta_L | L \text{ is an integral lattice of } V \text{ with } \text{level}(L) | N\}$$

For N and its positive divisor N' , define a subset $P(V; N, N')$ of \mathbf{h} by

$$P(V; N, N') = \left\{ p \in \mathbf{h} \mid N_p > 1, N'_p = 0, (-\sqrt{-1}^{n\delta_2(p)})_{\varepsilon_p(V)} = -1 \right\}$$

where we write the prime factorization of N, N' as

$$N = \prod_{p \in \mathbf{h}} p^{N_p}, \quad N' = \prod_{p \in \mathbf{h}} p^{N'_p},$$

($0 \leq N_p, N'_p \in \mathbb{Z}$) and set

$$\delta_2(p) = \begin{cases} 0 & p \neq 2 \\ 1 & p = 2 \end{cases}$$

for any $p \in \mathbf{h}$.

Since $\det V \equiv 1 \pmod{(\mathbb{Q}^\times)^2}$, we can write

$$4^n \det L = \prod_{p \in \mathbf{h}} p^{2t_p(L)}$$

($t_p(L) \in \mathbb{Z}$, $p \in \mathbf{h}$) for any lattice L of V .

For N and its positive divisor N' , define the global trace operator $T_{N,N'}^{(n)}$ by the following formula:

$$T_{N,N'}^{(n)}(f) = \sum_{\gamma \in \Gamma_0^{(m)}(N) \setminus \Gamma_0^{(m)}(N')} f|_n \gamma \quad (f \in \mathcal{M}_m(n, N)).$$

Then $T_{N,N'}^{(n)}$ is a well-defined \mathbb{C} -linear mapping of $\mathcal{M}_m(n, N)$ onto $\mathcal{M}_m(n, N')$.

4.2. Statement of main results. Now we state our main results.

Theorem V. *Let the notation and the assumptions be as above. Write the prime factorization of N, N' as*

$$N = \prod_{p \in \mathbf{h}} p^{N_p}, \quad N' = \prod_{p \in \mathbf{h}} p^{N'_p}.$$

Suppose

$$\begin{cases} 2 \leq N_2 = N'_2 & n \text{ odd,} \\ 2 \leq N_2 = N'_2 \text{ or } 1 \geq N_2 \geq N'_2 \geq 0 & n \text{ odd.} \end{cases}$$

Then we have

$$\Theta_m(V, N') = \Theta_m(V, N) \cap \mathcal{M}_m(n, N')$$

if $m \geq n$.

Theorem L. *Let the notation and the assumptions be as above. Suppose L is an integral lattice of V with $\text{level}(L) = N$. Set*

$$t_0(L) = \max \left\{ t_p(L) - \left\lfloor \frac{N_p - 1}{2} \right\rfloor \mid p \in P(V; N, N') \right\}$$

(1) *We have*

$$T_{N, N'}^{(n)}(\vartheta_L) \in \Theta_m(V, N')$$

if $m \geq \min\{n, t_o(L)\}$.

(2) *Furthermore suppose $t_o(L) > 0$. We obtain*

$$T_{N, N'}^{(n)}(\vartheta_L) = 0$$

if $m \geq \min\{n, t_o(L)\}$.

5. LOCALIZATION OF THE GLOBAL TRACES

5.1. Preliminaries. Let v be any place of \mathbb{Q} .

Define the nontrivial character $\psi_v : \mathbb{Q}_v \rightarrow \mathbb{T}$ by

$$\begin{aligned} \psi_v(x) &= \mathbf{e}(x) & \text{if } v = \infty, \\ \psi_v(x) &= \mathbf{e}(-\text{Fr}(x)) & \text{if } v \in \mathbf{h}, \end{aligned}$$

where $\text{Fr}(x)$ ($x \in \mathbb{Q}_p$) is the fractional part of the p -adic expansion of x .

Give a self-duality on X_v by $(x, y) \mapsto \psi_v(\text{tr}(2^t x S y))$.

For any $\varphi \in \mathcal{S}(X_v)$, define the Fourier transform $\widehat{\varphi}$ of φ by the following formula:

$$\widehat{\varphi}(y) = \int_{X_v} \varphi(x) \psi_v(\text{tr}(2^t x S y)) d_v x$$

($y \in X_v$). Here d_v is the self-dual Haar measure on X_v with respect to the above self-duality.

Under our assumptions, the Weil constant $\gamma_v(V)$ can be easily determined [2, 5]:

$$(5.1) \quad \gamma_v(V) = \begin{cases} (\sqrt{-1})^n & \text{if } v = \infty, \\ \varepsilon_v(V) & \text{if } v \in \mathbf{h} \setminus \{2\}, \\ (-\sqrt{-1})^n & \text{if } v = 2. \end{cases}$$

The extension $\mathbb{Q}_v(\sqrt{(-1)^n \det V})/\mathbb{Q}_v$ determines the unique character ω_v of \mathbb{Q}_v^\times by the local class field theory [6]: namely,

$$\omega_v(x) = (x, (-1)^n)_{\mathbb{Q}_v} \quad (x \in \mathbb{Q}_v^\times).$$

Notice that, for any $p \in \mathbf{h}$,

$$(5.2) \quad \ker \omega_p \supset \begin{cases} 1 + 4\mathbb{Z}_2 & p = 2, \\ \mathbb{Z}_p^\times & p \neq 2. \end{cases}$$

We have the so called local Weil representation π_v of G_v realized on $\mathcal{S}(X_v)$; π_v is characterized by the following three conditions (cf. [7]):

$$(5.3) \quad \left(\pi_v \left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right) \varphi \right) (x) = \psi_v(\mathrm{tr}(b^t x S x)) \varphi(x),$$

$$(5.4) \quad \left(\pi_v \left(\begin{bmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{bmatrix} \right) \varphi \right) (x) = \omega_v(\det a) |\det a|_v^n \varphi(xa),$$

$$(5.5) \quad \left(\pi_v \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \varphi \right) (x) = \gamma_v(V)^m \varphi^\wedge(x),$$

for $\varphi \in \mathcal{S}(X_v)$, $x \in X$, $a \in \mathrm{GL}_m(\mathbb{Q}_v)$ and $b \in \mathrm{Sym}_m(\mathbb{Q}_v)$. The mapping

$$G_v \times \mathcal{S}(X_v) \ni (g, \varphi) \mapsto \pi_v(g)\varphi \in \mathcal{S}(X_v)$$

is continuous. For any $p \in \mathbf{h}$ and $\varphi \in \mathcal{S}(X_p)$, the stabilizer of φ in G_p under π_p contains an open compact subgroup of G_p .

The global Weil representation $\pi_{\mathbf{A}}$ of $G_{\mathbf{A}}$ realized on $\mathcal{S}(X_{\mathbf{A}})$ is defined as follows. Let φ be an element of $\mathcal{S}(X_{\mathbf{A}})$ of the form $\varphi = \prod_v \varphi_v$ such that $\varphi_p = \mathbf{I}_{\mathrm{Mat}_{2n,m}(\mathbb{Z}_p)}$ for almost all

$p \in \mathfrak{h}$. For any $g = (g_v) \in G_{\mathbb{A}}$, put

$$\pi_{\mathbb{A}}(g)\varphi = \prod_v \pi_v(g_v)\varphi_v.$$

This action of $G_{\mathbb{A}}$ extends by continuity to the representation $\pi_{\mathbb{A}}$ of $G_{\mathbb{A}}$ on $\mathcal{S}(X_{\mathbb{A}})$.

Let v be any place of \mathbb{Q} . We give some examples of compact subgroups in G_v and of semi-invariant vectors under the action of these subgroups.

First, suppose $v = p \in \mathfrak{h}$. For any non-zero element λ of \mathbb{Z}_p , define an open compact subgroup $D_p(\lambda)$ of G_p by

$$D_p(\lambda) = \left\{ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_{\mathbb{Z}_p} \mid c \equiv 0 \pmod{\lambda\mathbb{Z}_p} \right\}.$$

If L_p be an integral lattice of V_p with $\text{lev}_p(L_p) = l$, then we can easily see $\ker \omega_p \supset (1 + p^l\mathbb{Z}_p) \cap \mathbb{Z}_p^\times$ (see (5.3),(5.4),(5.5)) and

$$(5.6) \quad \pi_p(g)I_{L_p^m} = \begin{cases} I_{L_p^m} & (l = 0) \\ \omega_p(\det a)I_{L_p^m} & (l \geq 1) \end{cases}$$

for any $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in D_p(p^l)$ (see (5.3),(5.4),(5.5), [9]).

Next suppose $v = \infty$. Set $\mathbf{i} = \sqrt{-1} \cdot 1_m \in \mathcal{H}_m$. Let U_∞ be the stabilizer of \mathbf{i} under the standard action of G_∞ on \mathcal{H}_m . We can immediately see

$$U_\infty = \left\{ u \in G_\infty \mid u = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \right\}$$

and that U_∞ is compact. For every $\mathfrak{z} \in \mathcal{H}_m$, we can define an element $\varphi_{\mathfrak{z}}$ of $\mathcal{S}(X_\infty)$ by

$$\varphi_{\mathfrak{z}}(x) = e(\text{tr}({}^t x S x \mathfrak{z}))$$

($x \in X_\infty$). Then we can show (cf. [9])

$$\pi_\infty(u)\varphi_{\mathbf{i}} = \det(A - \sqrt{-1}B)^n \varphi_{\mathbf{i}}$$

for any $u = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in U_\infty$. Therefore we obtain

$$(5.7) \quad \pi_\infty(g_\infty)\varphi_{\mathbf{i}} = j(g_\infty, \mathbf{i})^{-n} \varphi_{g_\infty \mathbf{i}}$$

for any $g \in G_\infty$.

5.2. The local trace operators. We define a character χ_n of a compact group $D_2(\lambda)$ ($\lambda \in \mathbb{Z}_2$ if n is even or $\lambda \in 4\mathbb{Z}_2 \setminus \{0\}$ if n is odd) by

$$(5.8) \quad \chi_n(g) = \begin{cases} 1 & \text{if } n \text{ even} \\ (\det a, -1)_{\mathbb{Q}_2} & \text{if } n \text{ odd} \end{cases}$$

$$\left(g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in D_2(\lambda) \right).$$

Take a non-negative integer l and a finite place $p \in \mathbf{h}$. Let $\mathcal{L}^{(m)}(V_p, l)$ be the subspace of $\mathcal{S}(X_p)$ spanned by the set of all functions of the form I_{M^m} such that M is an integral lattice of V_p with $\text{lev}_p(M) \leq l$. Notice that $\mathcal{L}^{(m)}(V_p, l) = 0$ if $p = 2$, n is odd and $0 \leq l \leq 1$, or if $p \in \mathbf{h}$, $\gamma_p(V_p) = -1$ and $l = 0$ (see (5.6)).

From now on, we assume that

$$(5.9) \quad \begin{cases} l \geq 0 & \text{if } p \neq 2, \\ l \geq 2 & \text{if } p = 2 \text{ and } n \text{ is odd,} \\ l \geq 0 & \text{if } p = 2 \text{ and } n \text{ is even.} \end{cases}$$

Under this assumption, we define a \mathbb{C} -linear map $\tau_{p,l}^{(m)}$ on $\mathcal{S}(X_p)$ by the following formula: (see (5.2), (5.6), (5.8), (5.9))

$$(5.10) \quad (\tau_{p,l}^{(m)}(\varphi))(x) = \begin{cases} \int_{D_p(p^l)} (\pi_p(u)\varphi)(x) d_{p,l}(u) & \text{if } p \neq 2, \\ \int_{D_p(p^l)} \chi_n(u) (\pi_p(u)\varphi)(x) d_{p,l}(u) & \text{if } p = 2, \end{cases}$$

$(\varphi \in \mathcal{S}(X_p), x \in X_p)$ where $d_{p,l}$ is the Haar measure of $D_p(p^l)$ normalized such that $\int_{D_p(p^l)} d_{p,l}(u) = 1$. The integrals on the right-hand side are essentially finite sum, and the map $\tau_{p,l}^{(m)}$ is well-defined. We call this map local trace operator.

By the definition of the local trace operator, we get $\tau_{p,l}^{(m)}(\varphi) = \varphi$ for $\varphi \in \mathcal{L}^{(m)}(V_p, l)$. Let l, l' be non-negative integers satisfying the above assumption and the condition $l \geq l' \geq 0$. The following theorem is a local analogue of Theorem L.

Theorem L_p . *Let the notation and assumption be as above. Take an integral lattice L_p of V_p with $\text{lev}_p(L_p) = l \geq 1$.*

(1) *Suppose $p \neq 2$ and $l > l' > 0$. If $m \geq 1$, we have*

$$\tau_{p,l'}^{(m)}(I_{L_p^{(m)}}) \in \mathcal{L}^{(m)}(V_p, l').$$

(2) *Suppose that $p \neq 2$ and $l > l' = 0$, or that $p = 2, l = 1 > l' = 0$ and n is even.*

(a) *If $\gamma_p(V_p) = 1$ and $m \geq 1$, we have*

$$\tau_{p,0}^{(m)}(I_{L_p^{(m)}}) \in \mathcal{L}^{(m)}(V_p, 0).$$

(b) *If $\gamma_p(V_p) = -1$ and $m \geq \min \left\{ n, t_p(L_p) - \left\lfloor \frac{l-1}{2} \right\rfloor \right\}$, we have*

$$\tau_{p,0}^{(m)}(I_{L_p^{(m)}}) = 0$$

and

$$\tau_{p,0}^{(m)}(I_{L_p^{(m)}}) \in \mathcal{L}^{(m)}(V_p, 0).$$

(c) *If $\gamma_p(V_p) = -1$ and $m < \min \left\{ n, t_p(L_p) - \left\lfloor \frac{l-1}{2} \right\rfloor \right\}$, we have*

$$\tau_{p,0}^{(m)}(I_{L_p^{(m)}}) \neq 0$$

and

$$\tau_{p,0}^{(m)}(I_{L_p^{(m)}}) \notin \mathcal{L}^{(m)}(V_p, 0).$$

Since $t_p(M) > t_p(M')$ for any two lattices M, M' of V_p with the condition $M \subsetneq M'$, Theorem L_p follows immediately from the two lemmas below.

Lemma 5.1. *The notation and the assumptions are as in Theorem L_p . Moreover suppose $p \neq 2$ and $l \geq 2$. Then, if $m \geq 1$, there exist a finite number of integral lattices $M_i (i \in I)$ of V_p satisfying*

$$l - 2 \leq \text{lev}_p(M_i) \leq l - 1 \quad (\forall i \in I)$$

such that

$$\tau_{p,l-1}^{(m)}(I_{L_p^m}) = \sum_{i \in I} c_i I_{M_i^m}$$

$(c_i \in \mathbb{C})$.

Lemma 5.2. *The notation and assumptions are as in theorem . Moreover, suppose $l = 1$.*

(1) *Let $\gamma_p(V_p) = 1$. Then, if $m \geq 1$, we obtain*

$$\tau_{p,0}^{(m)}(I_{L_p^m}) = c \sum_M I_{M^m}$$

$(c \in \mathbb{C}^\times)$, where the sum is taken over all integral lattices M of V_p such that $\text{lev}_p(M) = 0$ and $L_p \subset M \subset \check{L}_p$.

(2) *Let $\gamma_p(V_p) = -1$. Then, we have*

$$\begin{aligned} \tau_{p,0}^{(m)}(I_{L_p^m}) &= 0 & \text{if } m \geq \min \{n, t_p(L_p)\} \\ \tau_{p,0}^{(m)}(I_{L_p^m}) &\neq 0 & \text{if } m < \min \{n, t_p(L_p)\} \end{aligned}$$

We omit the proof of these two lemmas in this article.

5.3. Construction of theta series via the Weil representation. Let f be an element of $\mathcal{S}(X_h)$. Put

$$(\varphi_i \otimes f)(x) = \varphi_i(x_\infty) f(x_h) \quad x = (x_\infty, x_h) \in X_A,$$

then $\varphi_i \otimes f \in \mathcal{S}(X_A)$. For each $g \in G_A$, set

$$\Psi(f; g) = \sum_{x \in X_Q} \pi_A(g)(\varphi_i \otimes f)(x).$$

We can show (cf. [5, 8, 9])

- the series in the right-hand side is absolutely convergent on every compact subset of $G_{\mathbf{A}}$; hence $\Psi(f; \cdot)$ is a continuous function on $G_{\mathbf{A}}$;
- this function $\Psi(f; \cdot)$ is left $G_{\mathbf{Q}}$ invariant and right invariant for some open compact subgroup of $G_{\mathbf{h}}$;
- the restriction to G_{∞} determines $\Psi(f; \cdot)$ by the strong approximation theorem for $G_{\mathbf{A}}$.

For every $\mathfrak{z} \in \mathcal{H}_m$, take an element g_{∞} of G_{∞} such that $\mathfrak{z} = g_{\infty} \mathbf{i}$ and set

$$(5.11) \quad \vartheta(f; \mathfrak{z}) = j(g_{\infty}, \mathbf{i})^n \Psi(f; (g_{\infty}, 1_{\mathbf{h}})).$$

Then we have

$$\begin{aligned} \vartheta(f; \mathfrak{z}) &= \sum_{x \in X_{\mathbf{Q}}} (\varphi_{\mathfrak{z}} \otimes f)(x) \\ &= \sum_{x=(x_{\infty}, x_{\mathbf{h}}) \in X_{\mathbf{Q}}} \mathbf{e}(\mathrm{tr}({}^t x_{\infty} S x_{\infty} \mathfrak{z})) f(x_{\mathbf{h}}). \end{aligned}$$

Therefore we get a well-defined function $\mathfrak{z} \mapsto \vartheta(f; \mathfrak{z})$ on \mathcal{H}_m . Furthermore we can immediately see that

$$(5.12) \quad \vartheta(f; \gamma_{\infty} \mathfrak{z}) = j(\gamma_{\infty}, \mathbf{i})^n \vartheta(\pi_{\mathbf{h}}(\gamma_{\mathbf{h}}^{-1}) f; \mathfrak{z})$$

for any $\gamma = (\gamma_{\infty}, \gamma_{\mathbf{h}}) \in G_{\mathbf{Q}}$ and any $\mathfrak{z} \in \mathcal{H}_m$. Take an integral lattice L of V and put $f_{L^m} = \prod_{p \in \mathbf{h}} \mathbf{I}_{L_p^m}$. We can easily see $f_{L^m} \in \mathcal{S}(X_{\mathbf{h}})$ and

$$\vartheta(f_{L^m}; \cdot) = \vartheta_L.$$

5.4. Relation of the global and local traces. Let N be a positive integer and N' its positive divisor. We define an open compact subgroup $D_{\mathbf{h}}(N)$ of $G_{\mathbf{h}}$ by $D_{\mathbf{h}}(N) = \prod_{p \in \mathbf{h}} D_p(N)$ and set

$$D_{\mathbf{Q}}(N) = G_{\mathbf{Q}} \cap G_{\infty} D_{\mathbf{h}}(N).$$

By a morphism $\gamma \mapsto (\gamma, \gamma, \gamma, \dots)$, we identify $\Gamma_0^{(m)}(N)$ with $D_{\mathbf{Q}}(N)$. Since the strong approximation theorem holds for G , we can easily see that $G_{\infty} D_{\mathbf{Q}}(N)$ is dense in $G_{\infty} D_{\mathbf{h}}(N)$.

Thus the image of $D_{\mathbf{Q}}(N)$ under the canonical projection $G_{\mathbf{A}} \ni g = (g_{\infty}, g_{\mathbf{h}}) \mapsto g_{\mathbf{h}} \in$

$G_{\mathbf{h}}$ is dense in $D_{\mathbf{h}}(N)$. Therefore we can identify a complete set of representatives of $\Gamma_0^{(m)}(N) \setminus \Gamma_0^{(m)}(N')$ with that of $D_{\mathbf{h}}(N) \setminus D_{\mathbf{h}}(N')$.

From now on, assume (see (5.9))

$$(5.13) \quad \begin{cases} N'|N & \text{if } n \text{ even,} \\ 4|N'|N & \text{if } n \text{ odd.} \end{cases}$$

We see χ_n is also a character of $D_{\mathbf{h}}(N)$ via the canonical projection $D_{\mathbf{h}}(N) \rightarrow D_2(N)$. Let L be an integral lattice of V with $\text{level}(L)|N$. We compute the global trace $T_{N,N'}^{(n)}\vartheta_L$ as follows:

$$(5.14) \quad \begin{aligned} (T_{N,N'}^{(n)}\vartheta_L)(g_{\infty}\mathbf{i}) &= \sum_{\xi \in \Gamma_0^{(m)}(N) \setminus \Gamma_0^{(m)}(N')} \chi_n(\xi) j(\xi, g_{\infty}\mathbf{i})^{-n} \vartheta(f_{L^m}; \xi g_{\infty}\mathbf{i}) \\ &= j(g_{\infty}, \mathbf{i})^n \sum_{\gamma} \chi_n(\gamma_{\mathbf{h}}^{-1}) \Psi(f_{L^m}; (g_{\infty}, \gamma_{\mathbf{h}}^{-1})) \quad (\text{see (5.12)}) \end{aligned}$$

$$(5.15) \quad = \vartheta \left(\sum_u \chi_n(u) \pi_{\mathbf{h}}(u) f_{L^m}; (g_{\infty}, 1_{\mathbf{h}}) \right), \quad (\text{see (5.11)})$$

where $\gamma = (\gamma_{\infty}, \gamma_{\mathbf{h}})$ and u extend over $D_{\mathbf{Q}}(N) \setminus D_{\mathbf{Q}}(N')$ in (5.14) and $D_{\mathbf{h}}(N')/D_{\mathbf{h}}(N)$ in (5.15), respectively. Let the prime factorization of N' be $N' = \prod_{p \in \mathbf{h}} p^{l_p(N')}$. Notice that, since $D_p(N) = D_p(N')$ for almost all $p \in \mathbf{h}$, $\prod_{p \in \mathbf{h}} \tau_{p, l_p(N')}^{(m)}(I_{L_p^m})$ is an element of $\mathcal{S}(X_{\mathbf{h}})$. The last equality (5.15) shows that, up to the multiplication by an nonzero constant, the two functions

$$T_{N,N'}^{(n)}\vartheta_L \quad \text{and} \quad \vartheta \left(\prod_{p \in \mathbf{h}} \tau_{p, l_p(N')}^{(m)}(I_{L_p^m}); \cdot \right)$$

are equal on \mathcal{H}_m . Therefore Theorem L_p implies Theorem L.

REFERENCES

1. S. Böcherer. Traces on theta series. 保型形式シンポジウム報告書, pp. 101–105, 1993.
2. H. Jacquet and R.P. Langlands. *Automorphic forms on $GL(2)$* , volume 114 of *Lecture Note in Math.* Springer, 1970.
3. R. Salvati-Manni. Thetanullwerte and stable modular forms i. *Am.J.Math.*, Vol. 111, pp. 435–455, 1989.

4. R. Salvati-Manni. Thetanullwerte and stable modular forms ii. *Am.J.Math.* 113, Vol. 113, pp. 733–756, 1991.
5. A. Weil. Sur certains groupes d'opérateurs unitaires. *Acta math.*, Vol. 111, pp. 143–211, 1964.
6. A. Weil. *Basic Number Theory*. Springer, 1973.
7. H. Yoshida. Weil's representations of the symplectic groups over finite fields. *J.Math.Soc.*, Vol. 31, pp. 399–426, 1979.
8. H. Yoshida. Siegel modular forms and the arithmetic of quadratic forms. *Invent.math.*, Vol. 60, pp. 399–426, 1980.
9. H. Yoshida. On siegel modular forms obtained from theta series. *J.fur die reine und angew Math.*, Vol. 352, pp. 184–219, 1984.